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The k-hop connected dominating set problem: approximation and hardness

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Abstract Let G be a connected graph and k be a positive integer. A vertex subset D of G is a k-hop connected dominating set if the subgraph of G induced by D is connected, and for every vertex v in G there is a vertex u in D such that the distance between v and u in G is at most k. We study the problem of finding a minimum k-hop connected dominating set of a graph (MINk-CDS). We prove that MINk-CDS is $\mathcal{NP}$-hard on planar bipartite graphs of maximum degree 4. We also prove that MINk-CDS is $\mathcal{APX}$-complete on bipartite graphs of maximum degree 4. We present inapproximability thresholds for MINk-CDS on bipartite and on $(1, 2)$-split graphs. Interestingly, one of these thresholds is a parameter of the input graph which is not a function of its number of vertices. We also discuss the complexity of computing this graph parameter. On the positive side, we show an approximation algorithm for MINk-CDS. Finally, when $k = 1$, we present two new approximation algorithms for the weighted version of the problem restricted to graphs with a polynomially bounded number of minimal separators.

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1 Introduction

We address in this work generalizations of the minimum connected dominating set problem, a classic combinatorial optimization problem, from the perspective mainly of approximation algorithms and inapproximability results. We use standard notation and terminology in graph theory and approximation algorithms; for unexplained terms and symbols, the reader may refer to Ausiello et al. (1999) and Bondy and Murty (2008).

All graphs considered hereafter are assumed to be finite, simple and undirected. Let G be a graph. The sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. Throughout this paper, k denotes a positive integer. A set $D \subseteq V(G)$ is a *k-hop dominating set* (or k-DS, for short) of G if every vertex of G is within distance at most k from some vertex in D. If, additionally, $G[D]$ (the subgraph of G induced by D) is connected, then we call D a *k-hop connected dominating set* (or k-CDS, for short). If D is a 1-(C)DS, we simply say that D is a (connected) dominating set (or (C)DS, for short).

Let $\mathbb{Q}_>$ and $\mathbb{Z}_>$ be the sets of positive rational and positive integer numbers, respectively. For every vertex weight function $w\colon V(G) \to \mathbb{Q}_>$ and every non-empty subset $D \subseteq V(G)$, the *weight of D relative to w* (or the *weight of D*, when w is clear from context), denoted by $w(D)$, is defined as $\sum_{v\in D} w(v)$. For simplicity, we extend, in an analogous way, the notion of weight to non-empty subsets of any ground set. Henceforth, by vertex-weighted graph, we mean a graph whose vertices are assigned positive rational weights.

The *open k-neighborhood of a vertex v in G*, denoted by $N_G^k(v)$, is defined as the set of vertices in G, excluding v, that are at distance at most k from v. The *closed k-neighborhood of v in G*, denoted by $N_G^k[v]$, is defined as $\{v\} \cup N_G^k(v)$. For every non-empty subset $S \subseteq V(G)$, we define $N_G^k(S) := (\bigcup_{v\in S} N_G^k(v))\setminus S$ and $N_G^k[S] := \bigcup_{v\in S} N_G^k[v]$. To avoid notational overload, we omit the superscript k when $k = 1$ and the subscript G when the graph G is implicit.

The *minimum weight connected k-hop dominating set problem* (MINk-WCDS) asks for a minimum weight k-CDS of a given vertex-weighted graph. From this point on, whenever k appears in an acronym, as in MINk-CDS, or in a statement without explicit mention to its bounds, k should be read as *any fixed* positive integer. For convenience, we write MINk- CDS for the cardinality (unweighted) variants of the problem. In addition, if $k = 1$, then we write MIN(W)CDS.

A vertex $v \in V(G)$ is *k-universal* if $\{v\}$ is a k-CDS of G. We restrict MINk-WCDS only to graphs with at least $k + 2$ vertices, none of which is k-universal, and with maximum degree at least 3, since, otherwise, the problem can be solved efficiently.

In application scenarios involving ad hoc wireless networks, the design of so called virtual backbones is a central issue when it comes to the reduction of routing costs, signal interference and energy consumption. In such a context, k-hop connected dominating sets play an important role (Blum et al. 2005; Yu et al. 2013; Yadav et al. 2015).

2 Literature review

There is a vast literature on MinCDS and Min$k$-CDS. In this section, we mention some results which are closer to the aspects addressed in this paper, and note that this review is by no means exhaustive. To make the exposition clearer, we split this section into two subsections. The first one covers algorithms for polynomial-time solvable cases as well as approximation algorithms for MinCDS and Min$k$-CDS. The second one focuses on $\mathcal{NP}$-hardness and inapproximability results for these problems.

2.1 Polynomial cases and approximation algorithms

Many authors have devised efficient algorithms for MinCDS on restricted classes of graphs, such as distance-hereditary (D'Atri and Moscarini 1988), permutation (Colbourn and Stewart 1990), doubly chordal (Moscarini 1993), strongly chordal (White et al. 1985) and trapezoid graphs (Liang 1995). MinWCDS, that is, the weighted counterpart of MinCDS, is known to be solvable in polynomial time on series-parallel (White et al. 1985), interval (Ramalingam and Rangan 1988), distance-hereditary (Hong-Gwa and Chang 1998) and permutation graphs (Arvind and Regan 1992). Furthermore, Min$k$-CDS can be solved efficiently on distance-hereditary graphs (Brandstädt and Dragan 1998), HT-graphs (Dragan 1993) and graphs with bounded treewidth (Borradaile and Le 2015).

In regard to approximation algorithms for Min(W)CDS and Min$k$-CDS, we highlight the following works, with emphasis on results for arbitrary graphs.

Starting with MinCDS, Guha and Khuller (1998), to the best of our knowledge, gave the first contributions in this line of research, namely a greedy $2(H(\Delta(G))+1)$-approximation and a slightly more involved $(\ln \Delta(G) + 3)$-approximation, where G is the input graph and $H(\ell)$ denotes the ℓ-th Harmonic number. Six years later, Ruan et al. (2004) developed a $(\ln \Delta(G)+2)$-approximation, which was subsequently improved by Du et al. (2008), who proposed an approximation algorithm with ratio $(1+\epsilon)(1+\ln(\Delta(G)-1))$, for every fixed $0<\epsilon\leq 1$. Recently, Khuller and Yang (2016), building on some of the ideas of Guha and Khuller (1998), managed to devise two approximation algorithms for MinCDS, one with ratio $H(2\Delta(G)+1)+1$ and another with ratio $H(\Delta(G))+\sqrt{H(\Delta(G))}+1$. For cubic graphs, a $4/3$-approximation algorithm was given by Bonsma and Zickfeld (2008).

For MinWCDS, that is, when we allow vertices to have arbitrary positive rational weights, Guha and Khuller (1998) proposed a $3\ln n$-approximation for n-vertex graphs, which was then improved to a $(1.35+\epsilon)\ln n$-approximation, for every fixed $\varepsilon > 0$, by the same authors (Guha and Khuller 1999). A more comprehensive overview of the literature on approximation algorithms for Min(W)CDS can be found in Du and Wan (2012).

As far as we know, no approximation algorithms for Min$k$-CDS on general graphs have been proposed so far. However, Ren and Zhao (2011) presented an approximation algorithm for the *minimum connected set cover problem* (MinCSC), to which Min$k$-CDS can be reduced. In fact, these authors mention a reduction from MinCDS to MinCSC. We describe now how to extend this idea to reduce Min$k$-CDS to MinCSC.

In the MINCSC problem, we are given a triple $(U, \mathcal{C}, G)$, where U is a finite set (the universe), $\mathcal{C}$ is a family of subsets of U, and G is a graph with vertex set $\mathcal{C}$ (that is, the elements of $\mathcal{C}$ are the labels of the vertices of G). The objective of this problem is to find $K \subseteq \mathcal{C}$ with minimum cardinality such that $G[K]$ is connected and K covers U. To reduce an instance, say H, of MINk-CDS to an instance $(U, \mathcal{C}, G)$ of MINCSC, we proceed as follows. We let G be a graph isomorphic to H, take $U := V(H)$, and $\mathcal{C} := \{N_H^k[v] : v \in V(H)\}$; moreover, for each vertex $v \in V(H)$, its corresponding vertex in $V(G)$ receives the label $N_H^k[v]$ (the set of vertices in H at a distance at most k from v). Thus, given H, the construction of the instance for MINk-CDS consists basically in finding, for each vertex v, the set $N_H^k[v]$. Clearly, this can be done in polynomial time. The reader may refer to Ren and Zhao (2011) to see an illustration of this reduction for the case $k = 1$.

Ren and Zhao's algorithm for MINCSC outputs a $D_c(\mathcal{G})(H(\max_{X\in\mathcal{C}} |X|-1)+1)$-approximate solution, where $D_c(\mathcal{G})$ is the maximum distance in $\mathcal{G}$ taken over all pairs of vertices $X, Y \in \mathcal{C}$ such that $X \cap Y \neq \emptyset$. Note that Ren and Zhao's algorithm translates into a $2k(H(\Delta(G^k)) + 1)$-approximation for MINk-CDS where G^k denotes the *k-th power* of G, that is, the graph with vertex set $V(G)$ where two vertices are adjacent if they are within distance at most k in G.

Before closing this subsection, we should also point out that there is a substantial (and growing) body of works dealing with polynomial-time approximation schemes as well as decentralized (by that we mean distributed or local) approximations for MIN(W)CDS and MINk-CDS on special classes of graphs. For a selection of these papers, we refer the reader to Cheng et al. (2003), Nieberg and Hurink (2006), Zhang et al. (2009), Gao et al. (2010), Demaine and Hajiaghayi (2005), Cohen-Addad et al. (2016), Wan et al. (2002), Amiri et al. (2017), Dubhashi et al. (2005), Jallu et al. (2017) and other references therein.

2.2 $\mathcal{NP}$-hardness and inapproximability results

As for $\mathcal{NP}$-hardness results, MINCDS has been proven to be $\mathcal{NP}$-hard, for example, for split (White et al. 1985), chordal bipartite (Müller and Brandstädt 1987), and planar bipartite graphs (White et al. 1985). Moreover, Nguyen and Huynh (2006) showed that MINk-CDS is $\mathcal{NP}$-hard on planar unit disk graphs of maximum degree 4 and Lokshtanov et al. (2013) proved that MINk-CDS is $\mathcal{NP}$-hard on graphs with diameter $k + 1$. These are the strongest $\mathcal{NP}$-hardness results for MINk-CDS as far as we know. For further details on computational complexity results for MINCDS, we direct the reader to the book by Haynes et al. (1998) (although now somewhat outdated).

On the side of approximation hardness, in 2004, Chlebík and Chlebíková (2004) showed that, for every fixed $\varepsilon > 0$, there is no $(1 - \varepsilon)\ln n$-approximation algorithm for MINCDS on n-vertex bipartite and split graphs, unless $\mathcal{NP} \subseteq \mathrm{DTIME}(n^{O(\log\log n)})$. Four years later, the same authors (Chlebík and Chlebíková 2008) proved that there exist constants $C > 0$ and $B_0 \geq 3$ such that, for every $B \geq B_0$, it is $\mathcal{NP}$-hard to approximate MINCDS to within a factor of $\ln B - C \ln\ln B$ on bipartite graphs with maximum degree at most B. In 2012, Bonsma (2012) proved that MINCDS is $\mathcal{APX}$-

complete on cubic graphs. As far as we know, there have been no inapproximability results for MINk-CDS (that is, for general k) prior to this work.

3 Contributions

As discussed in the introduction, our contributions advance the state-of-the-art on MINk-CDS in two directions: approximation algorithms and inapproximability results.

As for approximations, we prove a type of meta-approximation theorem which says that, for every graph G, a $\beta(G)$-approximation for MINCDS on G can be turned into a $k\beta(G^k)$-approximation for MINk-CDS on G. As a consequence, we derive an algorithm that, for every fixed $0 < \varepsilon \leq 1$, finds a $k(1+\varepsilon)(1+\ln(\Delta(G^k)-1))$-approximation for MIN$k$-CDS on G, which is an improvement (asymptotically by a factor of 2) on Ren and Zhao's $2k(H(\Delta(G^k))+1)$-approximation (Ren and Zhao 2011).

We also propose two approximation algorithms for MINWCDS restricted to special classes of graphs (to be defined in Sect. 4), namely graphs with a polynomial number of minimal separators (which are inclusionwise minimal sets of vertices whose removal disconnects the graph). The first algorithm has an approximation factor which is logarithmic in the number of minimal separators of the input graph. Thus, we deem it more suitable for graphs with "few" separators (say with a linear or sublinear number of separators relative to the size of the graph, such as split graphs).

The second approximation algorithm has a performance guarantee that depends on a parameter of the input graph, namely the cardinality of its largest minimal separator, which is independent of its order (in the sense that this parameter does not necessarily grow with the number of vertices of the graph). Therefore, this second algorithm is more appropriate for graphs whose minimal separators contain a small number of vertices (say bounded by a constant). Interestingly, we show that, for certain classes of graphs, the approximation factor of the latter algorithm is close to the best one can hope for, assuming $\mathcal{P} \neq \mathcal{NP}$.

On the hardness side, in Sect. 5, we show that MINCDS is $\Omega(\log n)$-hard to approximate even on n-vertex split graphs with diameter 2, if $\mathcal{NP} \nsubseteq \mathrm{DTIME}(n^{O(\log\log n)})$. Asymptotically, this threshold is the same as the one proved by Chlebík and Chlebíková (2004) for MINCDS but it holds for the smaller class of split graphs of diameter 2 (surely the smallest value of the diameter for which the problem is non-trivial). In Sect. 6, we prove that MINk-CDS is $\mathcal{NP}$-hard on planar bipartite graphs of maximum degree 4. Moreover, we present inapproximability thresholds for MINk-CDS, generalizing the ones already known for MINCDS, on bipartite graphs and on a superclass of split graphs called $(1, 2)$-split graphs. Finally, we also show that MINk-CDS is $\mathcal{APX}$-complete on bipartite graphs of maximum degree 4. These results are shown to hold for every fixed k.

4 Approximation algorithms

For a problem Π and an instance I of Π, we denote by $\mathrm{OPT}_{\Pi}(I)$ the *value* (that is, the cardinality or the weight) of an optimal solution of Π for the instance I.

We show first an approximation algorithm for MinNk-CDS, and then we present two approximation algorithms for MinWCDS restricted to special classes of graphs. We start with the algorithm ApproxMin$k$- CDS (see Algorithm 1). It uses (as a subroutine) an approximation algorithm, say $\mathcal{A}$, for MinCDS. Given an input graph G, ApproxMin$k$- CDS computes G^k, and then it runs $\mathcal{A}$ on G^k, thus finding a CDS, say D, of G^k. Finally, it connects the components of $G[D]$ (in case $G[D]$ is disconnected) by adding at most $(k-1)(|D|-1)$ extra vertices to D.

Algorithm 1 ApproxMin$k$-CDS

Input: A graph G
Subroutine: An approximation algorithm $\mathcal{A}$ for MinCDS
Output: A k-CDS of G

1: Compute G^k $\triangleright$ the k-th power of G
2: $D \leftarrow \mathcal{A}(G^k)$ $\triangleright$ $\mathcal{A}$ is an approximation algorithm for MinCDS
3: $S \leftarrow D$
4: **while** $G[S]$ is not connected **do**
5: Choose vertices u and v in different components of $G[S]$ such that the distance between u and v in G is minimum
6: Compute a shortest path from u to v in G
7: Let P be the set of internal vertices of the path obtained in line 6
8: $S \leftarrow S \cup P$
9: **end while**
10: **return** S

Theorem 1 *Let G be a graph. If there exists an algorithm for* MinCDS *with approximation factor $\beta(G)$, then, for every $k \in \mathbb{Z}_>$, there exists an approximation algorithm for* Min$k$-CDS *with approximation factor $k\beta(G^k)$.*

Proof Let G be a graph and let $\mathcal{A}$ be a polynomial-time $\beta(G)$-approximation algorithm for MinCDS (used in step 2). Clearly, ApproxMin$k$- CDS runs in polynomial time. Moreover, it is immediate that it produces a k-CDS of G.

Let $S \subseteq V(G)$ be the solution output by ApproxMin$k$- CDS. Since a k-CDS of G is a CDS of G^k, it follows that $\mathrm{OPT}_{\mathrm{MinCDS}}(G^k) \leq \mathrm{OPT}_{\mathrm{Min}k\text{- CDS}}(G)$. Thus, it suffices to show that $|S| \leq k\beta(G^k)\mathrm{OPT}_{\mathrm{MinCDS}}(G^k)$. By hypothesis, we have that $|D| \leq \beta(G^k)\mathrm{OPT}_{\mathrm{MinCDS}}(G^k)$. If $G[D]$ is connected, then the proof is complete.

Assume now that $G[D]$ has at least two components. Let t be the number of iterations performed by the while loop in lines 4–9. Let $S_0 = D$ and, for every $j \in \{1, \ldots, t\}$, let S_j be the set S at the end of the jth iteration of the while loop. For every $j \in \{1, \ldots, t\}$, let u_j and v_j be the vertices chosen in the jth iteration in line 5, let C_{u_j} and C_{v_j} be the components of $G[S_{j-1}]$ that contain u_j and v_j, respectively, and let P_j be the set P chosen in line 7 in the jth iteration. Fix some $j \in \{1, \ldots, t\}$. Since $G^k[S_{j-1}]$ is connected, there is an edge of G^k with an endpoint, say w, in $V(C_{u_j})$ and an endpoint, say y, in some superset of $V(C_{v_j})$. Since y and w are adjacent in G^k, then y is within distance at most k from w in G and, thus, the same holds for u_j and v_j. Therefore, $|P_j| \leq k-1$ and $G[S_j]$ has at least one component less than $G[S_{j-1}]$. So we conclude that $t \leq |D|-1$. Furthermore, we have

$$|S| \leq |D| + (k-1)(|D|-1) \leq k|D|,$$

and the result follows. □

Du et al. (2008) proposed an approximation algorithm for MINCDS with ratio $(1+\epsilon)(1+\ln(\Delta(G)-1))$ for every fixed $0 < \epsilon \leq 1$. The following is a consequence of Theorem 1 using Du et al.'s algorithm as a subroutine.

Corollary 1 *For every $k \in \mathbb{Z}_>$ and every fixed $0 < \varepsilon \leq 1$, there exists a polynomial-time algorithm for* MINk-CDS *with approximation ratio $k(1+\epsilon)(1+\ln(\Delta(G^k)-1))$, where G is the input graph.*

As mentioned in the introduction, Ren and Zhao (2011) presented an approximation algorithm for MINCSC that, for every input graph G and every k, translates into a $2k(H(\Delta(G^k))+1)$-approximation for MINk-CDS. Since $\ln(\ell-1) < H(\ell)$ for every integer $\ell \geq 2$, Corollary 1 shows an improvement (asymptotically by a factor of 2) on Ren and Zhao's algorithm.

Observe that algorithm APPROXMINk-CDS indicates that MINk-CDS admits a constant approximation on bounded degree graphs. The approximation factor of this algorithm follows from the previous corollary and can be expressed in terms of k and the degree bound.

Corollary 2 *For every $k \in \mathbb{Z}_>$, there is a constant approximation algorithm for* MINk-CDS *on bounded degree graphs.*

We turn now our focus to MINWCDS. Before we describe the approximation algorithms for MINWCDS, we define some concepts. A *separator* of G is a subset $\Gamma \subseteq V(G)$ such that $G - \Gamma$ (the subgraph of G obtained from G by removing the vertices in Γ) has more components than G. A separator of G is *minimal* if it does not properly contain any other separator. Hereafter, the word minimal for a set means inclusionwise minimal.

A subset $\Gamma \subseteq V(G)$ is a *k-hop domination disruptive separator* (or *k-disruptive separator*, for short) if Γ is a separator of G and, for every component C of $G - \Gamma$, $V(C)$ is not a k-CDS of G. In other words, Γ is a k-disruptive separator of G if and only if Γ intersects every minimal k-CDS of G. Note that a vertex subset of G is a 1-disruptive separator of G if and only if it is a separator of G. We denote by $\mathcal{S}_k(G)$ (or simply $\mathcal{S}(G)$ when $k=1$) the set of all minimal k-disruptive separators of a graph G. Let $\sigma_k(G) = \max_{\Gamma \in \mathcal{S}_k(G)} |\Gamma|$ and we write $\sigma(G)$ when $k=1$.

Given a universe U and a collection $\mathcal{C}$ of subsets of U, a *transversal* of $\mathcal{C}$ is a subset of U that intersects every element of $\mathcal{C}$. The following result will be referenced many times.

Theorem 2 (Kanté et al. 2011) *Let G be a graph and $\mathcal{S}(G)$ be the set of all minimal separators of G. A set $D \subseteq V(G)$ is a CDS of G if and only if D is a transversal of $\mathcal{S}(G)$.*

We say that a class $\mathcal{C}$ of graphs *has a polynomial number of minimal k-disruptive separators* (or that $\mathcal{C}$ *has poly-k-separators*, for short) if there exists a polynomial p

such that $|\mathcal{S}_k(G)| \leq p(n)$ for every n-vertex graph G belonging to $\mathcal{C}$. We also say that a graph G *has poly-k-separators* if G belongs to some class with poly-k-separators. For simplicity, when $k = 1$, we write poly-separators to refer to poly-1-separators. Berry et al. (1999) designed an algorithm that, for every given graph G with n vertices, enumerates all minimal separators of G in $O(n^3)$ time per separator.

Many well-studied classes of graphs have poly-separators. Examples include chordal graphs (Chandran and Grandoni 2006), d-trapezoid graphs, weakly chordal graphs, co-comparability graphs with bounded dimension (Brandstädt et al. 1987), $2K_2$-free graphs (Dhanalakshmi et al. 2016) and P_4-sparse graphs (Nikolopoulos and Palios 2006) (for more examples, see Kloks and Kratsch 1998).

Now we discuss approximation algorithms for MinWCDS restricted to classes of graphs with poly-separators. The first one (see Algorithm 2) is CoverApprox MinWCDS. In order to explain how this algorithm functions, we have to define an auxiliary problem, namely the *minimum weight set cover problem* (MinWSC), which has the following description: given a universe U, a collection of subsets $\mathcal{C}$ of U and a weight function $w' : \mathcal{C} \to \mathbb{Q}_>$, the objective is to find a minimum weight cover $K \subseteq \mathcal{C}$ of U.

Essentially, CoverApproxMinWCDS works as follows: firstly, it takes an instance (G, w) of MinWCDS (where G is a graph with poly-separators and $w : V(G) \to \mathbb{Q}_>$ is a weight function) and "reduces" it to an "equivalent" instance of MinWSC; and, secondly, it runs on the resulting instance an approximation algorithm for MinWSC proposed by Chvátal (1979), denoted here as MinWSC- Chvátal.

Algorithm 2 CoverApproxMinWCDS

Input: A graph G with poly-separators and a weight function $w : V(G) \to \mathbb{Q}_>$
Subroutine: An approximation algorithm MinWSC- Chvátal for MinWSC
Output: A CDS of G

1: Run Berry et al.'s algorithm (1999) to compute $\mathcal{S}(G)$
2: **for** all $v \in V(G)$ **do**
3: $\quad\mathcal{F}_v \leftarrow \{\Gamma \in \mathcal{S}(G) : v \in \Gamma\}$
4: **end for**
5: $\mathcal{F} \leftarrow \{\mathcal{F}_v : v \in V(G)\}$
6: Let $w' : \mathcal{F} \to \mathbb{Q}_>$ be the weight function such that $w'(\mathcal{F}_v) = w(v)$ for every $v \in V(G)$
7: $K \leftarrow$ MinWSC- Chvátal$(\mathcal{S}(G), \mathcal{F}, w')$
8: $S \leftarrow \{v \in V(G) : \mathcal{F}_v \in K\}$
9: **return** S

Theorem 3 *Let G be a graph with poly-separators and $w : V(G) \to \mathbb{Q}_>$ a weight function. The algorithm* CoverApproxMinWCDS *produces, in polynomial time, an $H(|\mathcal{S}(G)|)$-approximate solution for* MinWCDS *on* (G, w).

Proof First, note that, since G has poly-separators, the algorithm runs in polynomial time. Let $S \subseteq V(G)$ be the solution output by the algorithm. By construction, S is a transversal of $\mathcal{S}(G)$, and thus, by Theorem 2, S is a CDS of G. As proved in Chvátal (1979), for any instance $(U, \mathcal{C}, w')$ of MinWSC, Chvátal's algorithm finds an $H(\ell)$-approximate cover of U, where ℓ is the cardinality of the largest set in $\mathcal{C}$. Therefore, algorithm CoverApproxMinWCDS yields an $H(|\mathcal{S}(G)|)$-approximate solution for MinWCDS on (G, w). □

We remark that, for n-vertex graphs with $o(n^{1.35})$ minimal separators (such as split graphs), CoverApproxMinWCDS may outperform Guha and Khuller's approximation (1999) for MinWCDS, which has a factor of $(1.35 + \varepsilon)\ln n$, for any fixed $\varepsilon > 0$.

Now we discuss one last approximation algorithm for MinWCDS, namely LPApproxMin- WCDS (see Algorithm 3). Recall that, for every graph G, we denote by $\sigma_k(G)$ the cardinality of the largest minimal k-disruptive separator of G.

In a conference version of this paper (Coelho et al. 2015), we stated a result (without proof), which says that, for every instance of MinWCDS, there exists an algorithm that efficiently finds a $\sigma(G)$-approximate solution for MinWCDS on an input graph G. Unfortunately, our argument was flawed because, in order to make it work, we needed a polynomial-time algorithm for computing $\sigma(G)$. In fact, we were able to prove that computing this graph parameter is an $\mathcal{NP}$-hard problem already for planar graphs with small maximum degree. For completeness, we include this proof here and then we discuss a $\sigma(G)$-approximation algorithm for MinWCDS for graphs with poly-separators.

Proposition 1 *For every $k \in \mathbb{Z}_>$, it is $\mathcal{NP}$-hard to compute $\sigma_k(G)$ even for planar bipartite graphs G with maximum degree* 4.

Proof The *maximum doubly connected cut problem* (MaxDCC) consists in finding in a connected graph G a *doubly connected cut* $(S, \overline{S})$ of maximum cardinality, that is, a partition $(S, \overline{S})$ of $V(G)$ such that $G[S]$ and $G[\overline{S}]$ are connected, and with the maximum number of *crossing edges* (edges with an endpoint in S and the other in $\overline{S}$).

Haglin and Venkatesan (1991) showed that the decision version of MaxDCC is $\mathcal{NP}$-complete on 3-connected cubic planar graphs. We present a reduction from MaxDCC to the decision version of our problem, which asks, given a graph G and an integer q, if G has a minimal separator containing at least q vertices.

Let G be a 3-connected cubic planar graph. We build from G a graph G' with vertex set $V(G') = V(G) \cup \{w_e\colon E(G)\} \cup (\cup_{v \in V(G)} V(P_v))$ as follows. First, we take G and subdivide each of its edges exactly once. For every $e \in E(G)$, we denote by w_e the vertex of the subdivision of edge e. For every v, we take a new path P_v (disjoint from the subdivision of G), with initial vertex u_v and with k vertices. Then, for every $v \in V(G)$, we add the edge $u_v v$ (that is, we append the path P_v to v). The graph G' has $|V(G)| + |E(G)| + k|V(G)|$ vertices. The construction of G' is depicted in Fig. 1. Clearly, G' can be constructed in time polynomial in the size of G. Furthermore, G' is planar, bipartite and it has maximum degree 4.

Observe that each $v \in V(G)$ is a cut vertex of G'. Thus every minimal k-disruptive separator in G' that contains any vertex of $V(G)$ has size exactly 1. Moreover, for every $v \in V(G)$, it is clear that no minimal k-CDS of G' intersects $V(P_v)$. Consequently, every minimal k-disruptive separator of G' of size larger than 1 is contained in $\{w_e\colon e \in E(G)\}$. It is easy now to see that $F = (S, \overline{S})$ is a doubly connected cut in G if and only if $\{w_f \in V(G')\colon f \in E_F\}$ is a minimal k-disruptive separator of G', where E_F is the set of crossing edges induced by F. Therefore, G has a doubly connected cut with at least q crossing edges if and only if G' has a minimal k-disruptive separator of size at least q. □

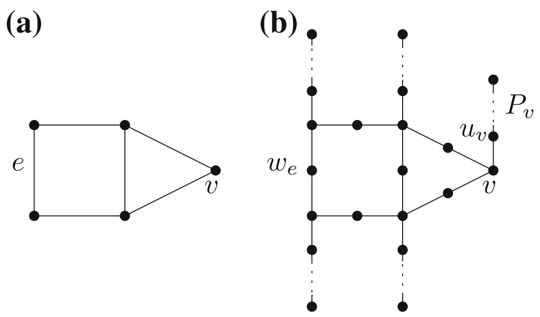

Fig. 1 Graph G' obtained from G with the construction described in the proof of Proposition 1. **a** Graph G. **b** Graph G'

Algorithm 3 LPApproxMinWCDS

Input: A graph G with poly-separators and a weight function $w : V(G) \to \mathbb{Q}_>$
Output: A CDS of G
1: Run Berry et al.'s algorithm (1999) to compute $\mathcal{S}(G)$
2: Let $x^* = (x^*_v)_{v \in V(G)}$ be an optimal solution to the linear program
$\min \left\{ \sum_{v \in V(G)} w(v)x_v : \sum_{v \in \Gamma} x_v \geq 1 \text{ for all } \Gamma \in \mathcal{S}(G), x_v \geq 0 \text{ for all } v \in V(G) \right\}$
3: Compute $\sigma(G)$
4: $D \leftarrow \{v \in V(G) : x^*_v \geq 1/\sigma(G)\}$
5: **return** D

Theorem 4 *Let G be a graph with poly-separators and $w : V(G) \to \mathbb{Q}_>$ a weight function. The algorithm* LPApproxMinWCDS *finds, in polynomial time, a $\sigma(G)$-approximate solution for* MinWCDS *on* (G, w).

Proof We claim that the set D output by the algorithm is a CDS of G. Let x^* be as defined in line 2. For every $\Gamma \in \mathcal{S}(G)$, we have that $\max\{x^*_v : v \in \Gamma\} \geq 1/|\Gamma| \geq 1/\sigma(G)$. Thus, D is a transversal of $\mathcal{S}(G)$, and by Theorem 2, D is a CDS of G. Again, by Theorem 2, we know that every CDS of G yields a feasible solution to the linear program defined in line 2. Thus, $\sum_{v \in V(G)} w(v)x^*_v \leq \mathrm{OPT}_{\mathrm{MinWCDS}}(G, w)$, and therefore,

$$w(D) = \sum_{v \in D} w(v) \leq \sigma(G) \left(\sum_{v \in D} w(v)x^*_v \right) \leq \sigma(G) \cdot \mathrm{OPT}_{\mathrm{MinWCDS}}(G, w).$$

Since G has poly-separators, LPApproxMinWCDS runs in polynomial time. This concludes the proof. □

We discuss now why we were not able to extend Theorem 4 to Min$k$-WCDS. The crux of the matter here is that, for some graphs G, it seems that the structure of $\mathcal{S}_k(G)$, when $k \geq 2$, is quite different (and much more complex, we think) when compared to $\mathcal{S}(G)$. We are not aware of any algorithm that enumerates, for *every* $k \geq 2$, all minimal k-disruptive separators of *any* given input graph G *with poly-k-separators* in polynomial time in the size of G. In fact, we would like to posit the following conjecture.

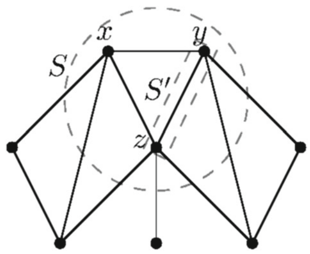

Fig. 2 The vertex subset $S = \{x, y, z\}$ is a minimal 2-disruptive separator but it is not a minimal separator

Conjecture 1 There exists an integer $k \geq 2$ and a class $\mathcal{C}$ of graphs with poly-k-separators for which it is $\mathcal{NP}$-hard to compute $\sigma_k(G)$ for every graph G belonging to $\mathcal{C}$.

Observe that, if $\mathcal{P} \neq \mathcal{NP}$ and if Conjecture 1 is true, then there cannot be an algorithm that, for *all* $k \geq 2$ and *every* n-vertex graph with poly-k-separators, lists the elements of $\mathcal{S}_k(G)$ in $n^{O(1)}$ time. Note that, for $k = 1$, there *is* such an algorithm, namely Berry et al.'s algorithm (1999). Thus, Proposition 1 implies (for $k = 1$), assuming $\mathcal{P} \neq \mathcal{NP}$, that not all bipartite planar graphs with maximum degree 4 have poly-separators. Otherwise, using Berry et al.'s algorithm, we could compute $\sigma(G)$ efficiently for graphs G in this class, implying that $\mathcal{P} = \mathcal{NP}$.

Another question related to the graph parameter σ_k concerns the relation (if any) between the sets $\mathcal{S}_k(G)$, $\mathcal{S}(G)$ and $\mathcal{S}(G^k)$ for every graph G. At first, we attempted to prove that $\mathcal{S}_k(G) \subseteq \mathcal{S}(G) \cup \mathcal{S}(G^k)$. However, as illustrated in Fig. 2, already for $k = 2$, such inclusion does not hold. It is straightforward to check, in Fig. 2, that the set of vertices $S = \{x, y, z\}$ is a minimal 2-disruptive separator and *not* a minimal separator because it properly contains a separator, for instance $S' = \{y, z\}$. Moreover, one can check that the vertices x, y, z do not form a separator in the square of the graph depicted in Fig. 2. It would be interesting to find out, for general graphs or even for particular classes of graphs, how the set of all minimal k-disruptive separators, when $k \geq 2$, relates to other graph parameters.

5 Hardness results for MinCDS

In this section, we strengthen some known results in the MinCDS literature concerning hardness of approximation. Before we state the results, we recall some definitions. For any graph G, we denote by $\alpha(G)$ and $\omega(G)$ the stability number (the size of the largest stable set) and the clique number (the size of the largest clique) of G, respectively. We say that G is a *split graph* if $V(G)$ can be partitioned into two sets K and S such that K is a clique and S is a stable set in G. Such a partition (K, S) is called a *split partition* of G. We point out, for future reference, that Heggernes and Kratsch (2007) designed a linear-time algorithm (henceforth called Heggernes–Kratsch algorithm) that finds a split partition for any given split graph.

We now discuss the main results of this section. Firstly, we show that MinCDS is $\Omega(\log n)$-hard to approximate even on n-vertex split graphs with diameter 2, assuming $\mathcal{NP} \not\subseteq \mathrm{DTIME}(n^{O(\log \log n)})$. Lokshtanov et al. (2013) proved that MinCDS is $\mathcal{NP}$-hard on split graphs with diameter 2. In the sequence, we show that the polynomial-time reduction discussed in Lokshtanov et al. (2013) can be used to prove hardness of approximation for MinCDS on split graphs with diameter 2. Before proving that, we state two theorems, for future reference, and a support lemma.

Theorem 5 (Chlebík and Chlebíková 2004) *For every fixed* $\epsilon > 0$, MinCDS *cannot be approximated to within a factor of* $(1-\varepsilon)\ln n$ *on* n*-vertex split graphs and on bipartite graphs, unless* $\mathcal{NP} \subseteq \mathrm{DTIME}(n^{O(\log\log n)})$.

Theorem 6 (Golumbic 2004, Theorem 6.2) *Let* G *be a split graph and let* (K, S) *be a split partition of* G*. If* K *is a maximum clique and* S *is a maximum stable set in* G*, then* (K, S) *is the unique split partition of* G.

Lemma 1 *For every fixed* $\epsilon > 0$, MinCDS *cannot be approximated to within a factor of* $(1-\varepsilon)\ln n$ *on* n*-vertex split graphs with a unique split partition having both sides of even cardinality, unless* $\mathcal{NP} \subseteq \mathrm{DTIME}(n^{O(\log\log n)})$.

Proof In what follows, we describe a polynomial-time reduction that, for every given split graph G, produces a split graph G' with a unique split partition having both sides of even cardinality.

The reduction goes as follows. Let G be a split graph. We run the Heggernes–Kratsch algorithm on G to obtain a split partition (K, S) of G. Let $p = |K|$ and $\ell = |S|$ and suppose that $K = \{c_1, \ldots, c_p\}$ and $S := \{s_1, \ldots, s_\ell\}$. It is straightforward to see that we may assume that S is a maximal stable set in G. Let G' be a disjoint copy of G. For each $v \in V(G)$, we denote by v' the copy of v in G'. Consider the natural split partition $K' = \{v' : v \in K\}$ and $S' = \{v' : v \in S\}$ of G'. We now define a graph H obtained from the union of G and G' by adding all possible edges between vertices in K and K'. Clearly, $(K \cup K', S \cup S')$ is a split partition of H, and $\alpha(H) = |S \cup S'|$ and $\omega(H) = |K \cup K'|$. Hence, by Theorem 6, H has a unique split partition.

It is easy to prove that if G has a CDS with at most q vertices then H has a CDS with at most $2q$ vertices. The converse also holds. Suppose H has CDS $\hat{D}$ such that $|\hat{D}| \leq 2q$. We may assume that $\hat{D} \subseteq K \cup K'$. We can also suppose, without loss of generality, that $|\hat{D} \cap K| \leq |\hat{D} \cap K'|$. Since $\hat{D}$ is a CDS of H, then, by the construction of H, we conclude that $\hat{D} \cap K$ is a CDS of G with at most q vertices.

Thus, we conclude that $\mathrm{OPT}_{\mathrm{MinCDS}}(H) \leq 2\mathrm{OPT}_{\mathrm{MinCDS}}(G)$. Now we are ready to prove the lemma. Assume $\mathcal{NP} \nsubseteq \mathrm{DTIME}(n^{O(\log\log n)})$ and suppose, for a contradiction, that there exists an approximation algorithm with ratio $(1-\varepsilon)\ln n$, where $\varepsilon < 1$ is a fixed positive constant, for MinCDS on n-vertex split graphs with a unique split partition having both sides of even cardinality. Let us call such algorithm $\mathcal{A}_\varepsilon$. Consider the following algorithm $\mathcal{A}'$ that, for every n-vertex split graph G, runs as follows.

Step 1. Check if $n^\varepsilon < 2$. If yes, then solve MinCDS on G by brute force and return an optimal solution. Otherwise, go to the next step;
Step 2. Run the reduction described previously on G to obtain H;
Step 3. Run $\mathcal{A}_\varepsilon$ on H to obtain $\hat{D}$;
Step 4. Compute D from $\hat{D}$ (as explained in the reduction) and return D.

One may easily check that $\mathcal{A}'$ is a polynomial-time algorithm. We claim that $\mathcal{A}'$ always returns a $(1-\varepsilon^2)\ln n$-approximate CDS of G. Indeed, if $\mathcal{A}'$ halts on step 1,

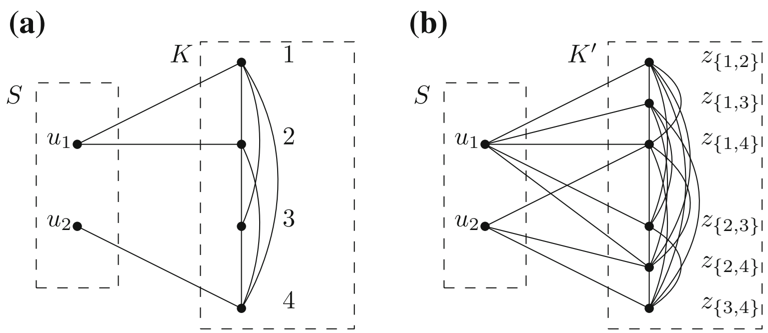

Fig. 3 Reduction described in the proof of Theorem 7. **a** Graph G with split partition (K, S). **b** Graph G' with diameter 2 and split partition (K', S)

then, by construction, it returns an optimal CDS of G. Suppose that $\mathcal{A}'$ halts after step 4. By hypothesis, $|\hat{D}| \leq (1-\varepsilon)\ln|V(H)|\text{OPT}_{\text{MINCDS}}(H)$. But now, since $|\hat{D}| \geq 2|D|$, $\text{OPT}_{\text{MINCDS}}(H) \leq 2\text{OPT}_{\text{MINCDS}}(G)$, $|V(H)| = 2n$ and $n^{\varepsilon} \geq 2$, we conclude that

$$|D| \leq \left((1-\varepsilon)\ln n^{1+\varepsilon}\right) \text{OPT}_{\text{MINCDS}}(G) = \left(\left(1-\varepsilon^2\right)\ln n\right) \text{OPT}_{\text{MINCDS}}(G).$$

Therefore, the existence of $\mathcal{A}'$ contradicts Theorem 5, and the result follows. □

The next result indicates that the inapproximability threshold for MINCDS in Theorem 5, proven by Chlebík and Chlebíková (2004), remains unchanged, asymptotically speaking, even if we restrict the problem to graphs of diameter 2. We note that the reduction constructed in the proof of Chlebík and Chlebíková's result produces graphs with diameter greater than 2.

Theorem 7 MINCDS *cannot be approximated to within a factor of $c \ln n$ for any positive constant $c < 1/4$ on n-vertex split graphs with diameter 2, unless $\mathcal{NP} \subseteq$* DTIME$(n^{O(\log\log n)})$.

Proof Let us first recall the reduction constructed by Lokshtanov et al. (2013) to show that MINCDS is $\mathcal{NP}$-hard on split graphs with diameter 2.

Given a split graph G with a unique split partition having both sides of even cardinality, first we run the Heggernes–Kratsch algorithm to find the split partition (K, S) of $V(G)$. We define a graph G' with vertex set $V(G') = S \cup K'$, where $K' := \{z_e : e \in E(G[K])\}$. For every edge $e = uv \in E(G[K])$, G' has an edge $z_e w$ for every $w \in (N_G(u) \cup N_G(v)) \cap S$. Moreover, G' has all possible edges between vertices belonging to K', that is, K' induces a clique in G'. This construction is depicted in Fig. 3. Clearly, G' is a split graph with diameter 2 and can be constructed in time polynomial in the size of G. Furthermore, observe that $|V(G')| \leq |V(G)|^2$.

We prove first that, for every CDS D of G with $|D| = q$, we can find in polynomial time a CDS D' of G' with $|D'| \leq (q+1)/2$. We may assume that $D \subseteq K$. Note that if $|D|$ is odd, then D is properly contained in K because, by hypothesis, $|K|$ is even. Consider $\widetilde{D} \subseteq V(G)$ defined as follows: let $\widetilde{D} = D$ if $|D|$ is even and let $\widetilde{D} = D \cup \{v\}$

otherwise, where $v \in K \backslash D$. Note that, by construction, $|\widetilde{D}|$ is even. Let $s = |\widetilde{D}|$ and suppose that $\widetilde{D} = \{v_1, \ldots, v_s\}$. Let $D' := \{z_{\{v_i, v_{i+s/2}\}} \in V(G') : i = 1, \ldots, s/2\}$. Observe that $|D'| \le (q+1)/2$. Since D is a CDS of G, then, by the construction of G', it is straightforward to check that D' is a CDS of G'. Thus, we conclude that, for every CDS of G with size q, we can find, in polynomial time, a CDS of G' with size at most $(q+1)/2$. Thus, we conclude that $\text{OPT}_{\text{MINCDS}}(G') \le \text{OPT}_{\text{MINCDS}}(G)$.

Now we prove that, for every CDS D' of G' with $|D'| = q$, we can find in polynomial time a CDS D of G such that $|D| \le 2q$. We may assume that $D' \subseteq K'$. In this case, the set $D = \{u, v \in V(G) : z_{\{u,v\}} \in D'\}$ has the desired properties.

Thus, given an $\alpha \ln |V(G')|$-approximate CDS of G', where α is some positive constant, we can find, in polynomial time, a $4\alpha \ln |V(G)|$-approximate CDS of G.

Now we are ready to conclude the proof of the theorem. Assume that $\mathcal{NP} \not\subseteq \text{DTIME}(n^{O(\log\log n)})$, and suppose there exists an approximation algorithm with ratio $c \ln n$, for some constant $c < 1/4$, for MINCDS on n-vertex split graphs with diameter 2. Let us call such algorithm $\mathcal{A}_c$. Let $\mathcal{A}'$ be the algorithm for MINCDS on the class of graphs G described in Lemma 1 defined as follows: given G, it constructs G', as we mentioned previously, and runs $\mathcal{A}_c$ on G'. Thus, for $\varepsilon \le 1 - 4c < 1$, algorithm $\mathcal{A}'$ obtains a $(1-\varepsilon)\ln n$-approximate CDS of G, a contradiction to Lemma 1. □

6 Hardness results for MIN*k*-CDS

In this section we address complexity issues on MINk-CDS from the standpoint of finding exact or approximate solutions.

As mentioned in the introduction, Nguyen and Huynh (2006) showed that MINk-CDS is $\mathcal{NP}$-hard on planar unit disk graphs with maximum degree 4. Their proof, which is quite involved, clearly implies that MINk-CDS is $\mathcal{NP}$-hard on planar graphs with maximum degree 4. The next result, with a simpler proof, has the same implication. Furthermore, it strengthens and generalizes a theorem of White et al. (1985) who showed that MINCDS is $\mathcal{NP}$-hard on planar bipartite graphs.

Theorem 8 *For every $k \in \mathbb{Z}_>$, the decision version of* MINk-CDS *is $\mathcal{NP}$-complete on planar bipartite graphs of maximum degree* 4.

Proof We say that a vertex subset K of a graph G is a *connected vertex cover* (CVC) of G if every edge in G has at least one endpoint in K and $G[K]$ is connected. The *minimum connected vertex cover problem* (MINCVC) seeks for a CVC of G of minimum cardinality.

Fernau and Manlove (2009) proved that the decision version of MINCVC is $\mathcal{NP}$-complete on planar bipartite graphs of maximum degree 4. We present a polynomial-time reduction from the decision version of MINCVC to the decision version of MINk-CDS, which is clearly a problem in $\mathcal{NP}$.

Let G be a planar bipartite graph with maximum degree 4. For each edge $e \in E(G)$ with endpoints u and v, we remove e from G, take a disjoint path P_e with $k+1$ new vertices, and then add edges $w_e u$ and $w_e v$, where w_e is an endpoint of P_e. Let G' be the graph obtained from G with this procedure. Clearly, G' is planar bipartite, and has maximum degree 4. The reduction is depicted in Fig. 4.

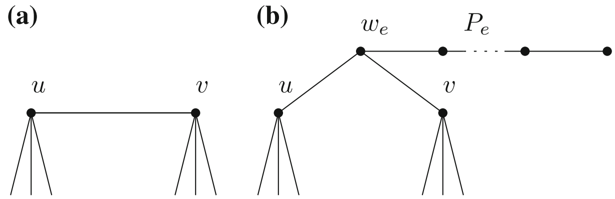

Fig. 4 Gadget described in the proof of Theorem 8. **a** An edge $e = uv$ of graph G. **b** P_e is a path with $k+1$ vertices and endpoint w_e

We claim that G has a CVC of size at most q if and only if G' has a k-CDS of size at most $|E(G)| + q$.

Let K be a CVC of G with $|K| \leq q$. Take $D = K \cup \{w_e : e \in E(G)\}$. Clearly, the size of D is at most $|E(G)| + q$. Since K induces a connected subgraph of G, it follows that D induces a connected subgraph of G'. For every $e \in E(G)$, the distance between each vertex of P_e and its endpoint w_e is at most k. Consequently, D is a k-CDS of G' such that $|D| \leq |E(G)| + q$.

Let D' be a k-CDS of G' with $|D'| \leq |E(G)| + q$. Take $K = D' \cap V(G)$. Since, for every $e \in E(G)$, the path P_e in G' has $k+1$ vertices, we conclude that $w_e \in D'$, and thus $|D' \cap (V(G') \backslash V(G))| \geq |E(G)|$. Consequently, $|K| \leq q$. Since $w_e \in D'$ for every $e \in E(G)$ and D' induces a connected subgraph of G', we conclude that K is a vertex cover of G and $G[K]$ is connected. Therefore, K is a CVC of G, and this completes the proof. $\square$

Let p, q be positive integers. We say that a graph G is (p, q)-*split* if there is a partition (K, S) of $V(G)$ such that $\alpha(G[K]) \leq p$ and $\omega(G[S]) \leq q$. We call such a partition a (p, q)-split partition. Note that every split graph is contained in the class of (p, q)-split graphs, since every split graph is a $(1, 1)$-split graph (see Gyárfás 1998).

We present now an inapproximability result for MINk-CDS that can be seen as a generalization of Theorem 5. First, we observe that every split graph has a 2-universal vertex (just take any vertex in the clique side of a split partition). Thus, MINk-CDS becomes trivial on split graphs for every $k \geq 2$, and therefore one can only obtain a result like Theorem 9 for a superclass of split graphs. We were able to prove an inapproximabilty result for the class of $(1, 2)$-split graphs, as we show in what follows.

Theorem 9 *For every $k \in \mathbb{Z}_>$ and every fixed $\varepsilon > 0$,* MINk-CDS *cannot be approximated to within a factor of* $(1 - \varepsilon) \ln n$ *on* n*-vertex* $(1, 2)$*-split graphs and on bipartite graphs, unless* $\mathcal{NP} \subseteq \mathrm{DTIME}(n^{O(\log \log n)})$

Proof We first prove the claimed inapproximability threshold for bipartite graphs. We next show a reduction from MINCDS on split graphs to MINk-CDS on bipartite graphs. Naturally, we assume that $k \geq 2$ because the result is already proven for $k = 1$.

Let G be a split graph. Firstly, we run the Heggernes–Kratsch algorithm on G and obtain a split partition (K, S) of G. Let G' be the graph obtained from G as follows. For every $v \in S$, take a disjoint path P_v with $k - 1$ new vertices and endpoint v', and add an edge connecting v to v'. Additionally, take another disjoint path P_h with $k+1$ new vertices and endpoint h. Then, for each $v \in K$, add an edge connecting v to h.

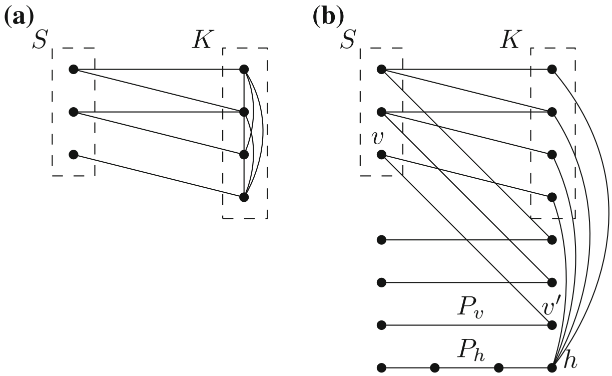

Fig. 5 Reduction described in the proof of Theorem 9 (bipartite case). **a** Graph G with split partition (K, S). **b** Bipartite graph G' obtained from G when $k = 3$

Finally, remove all edges with both endpoints in K. We depict the construction of G' from G in Fig. 5. Clearly, G' is bipartite and can be constructed in time polynomial in the size of G. Furthermore, observe that $|V(G')| \leq k|V(G)| + k + 1$.

We claim that G has a CDS of size at most q if and only if G' has a k-CDS of size at most $q + 1$. Let D be a CDS of G such that $|D| \leq q$. We may assume, without loss of generality, that $D \subseteq K$. Let us define $D' = D \cup \{h\}$. Since D is a CDS in G, then D' is a k-CDS in G'. Hence, for every CDS of G with size at most q, we can find, in polynomial time, a k-CDS of G' with size at most $q + 1$.

Consider now a k-CDS D' of G' such that $|D'| \leq q + 1$. Since D' induces a connected subgraph of G' and $|V(P_h)| = k + 1$, we conclude that $h \in D'$. Let us define $D = D' \cap K$. Since D' is a k-CDS of G', one may easily verify that D is a CDS of G and $|D| \leq q$. Hence, for every k-CDS of G' with size at most $q + 1$, we can find, in polynomial time, a CDS of G with size at most q. Therefore, we conclude that $\mathrm{OPT}_{\text{MIN}k\text{-CDS}}(G') = \mathrm{OPT}_{\text{MINCDS}}(G) + 1$.

Assume $\mathcal{NP} \nsubseteq \mathrm{DTIME}(n^{O(\log\log n)})$ and suppose, to the contrary, that there exists an approximation algorithm for MINk-CDS on n-vertex bipartite graphs with ratio $(1-\varepsilon)\ln n$, where $\varepsilon < 1$ is some fixed positive constant. Let us call such algorithm $\mathcal{A}_\varepsilon$. Consider the following algorithm $\mathcal{A}'$ that, for every input n-vertex split graph G runs as follows.

Step 1. Check if $n < k+2$. If yes, then solve MINCDS on G by brute force. Otherwise, go to the next step;

Step 2. Check if $\mathrm{OPT}_{\text{MINCDS}}(G) < 1/\varepsilon$ (by enumerating, via brute force, all possible solutions, if any, with at most $\lfloor 1/\varepsilon \rfloor$ vertices; remember that ε is a fixed constant). If yes, then solve MINCDS on G by brute force. Otherwise, go to the next step;

Step 3. Check if $n^{\varepsilon^2} < (k + 1)$. If yes, then solve MINCDS on G by brute force. Otherwise, go to the next step;

Step 4. Run the reduction described previously on G to obtain G';

Step 5. Run $\mathcal{A}_\varepsilon$ on G' to obtain D';

Step 6. Compute D from D' (as explained in the reduction) and return D.

It is immediate that $\mathcal{A}'$ is a polynomial time algorithm. Now we claim that $\mathcal{A}'$ always find a $(1-\varepsilon^4)\ln n$-approximate CDS of G. If $\mathcal{A}'$ halts before step 4, then it returns an optimal CDS of G. Suppose now that $\mathcal{A}'$ halts after step 6. By hypothesis, we have that $|D'| \leq (1-\varepsilon)\ln|V(G')|\mathrm{OPT}_{\mathrm{MinCDS}}(G')$. But since $|D'| > |D|, n \geq k+2$, $\mathrm{OPT}_{\mathrm{MinCDS}}(G) \geq 1/\varepsilon$ and $n^{\varepsilon^2} \geq (k+1)$, we conclude that

$$|D| \leq (1-\varepsilon)\ln n^{1+\varepsilon^2}(1+\varepsilon)\mathrm{OPT}_{\mathrm{MinCDS}}(G) = \left(1-\varepsilon^4\right)\ln n\mathrm{OPT}_{\mathrm{MinCDS}}(G).$$

Therefore, the existence of $\mathcal{A}'$ contradicts Theorem 5, and the result follows.

Now we prove the result for (1, 2)-split graphs. Since the reduction for (1, 2)-split graphs is quite similar to the one we discussed for bipartite graphs, we present only a sketch of the proof. The idea is to show a reduction from MinCDS on split graphs to Min$k$-CDS on (1, 2)-split graphs. Consider a split graph G. As before, we begin by running the Heggernes–Kratsch algorithm on G and we obtain a split partition (K, S) of G.

Let G' be the graph obtained from G as follows. For every vertex $v \in S$, we replace it by a path P_v with k vertices in such a way that vertex v is identified with an endpoint of P_v. We keep all other vertices and edges of G intact. In summary, $V(G') = K \cup (\cup_{v\in S} V(P_v))$ and $E(G') = E(G) \cup (\cup_{v\in S} E(P_v))$. Furthermore, note that $|V(G')| \leq k|V(G)|$. Note also that $\alpha(G'[K]) = 1$ and $\omega(G'[V(G')\backslash K]) = 2$. Thus, G' is a (1, 2)-split graph. Moreover, it can be constructed in time polynomial in the size of G.

It is not hard to see that G has a CDS of size at most q if and only if G' has a k-CDS of size at most q. Therefore, $\mathrm{OPT}_{\mathrm{Min}k\text{-CDS}}(G') = \mathrm{OPT}_{\mathrm{MinCDS}}(G)$.

Assume that there exists an approximation algorithm for Min$k$-CDS on n-vertex (1, 2)-split graphs with ratio $(1-\varepsilon)\ln n$, where $\varepsilon < 1$ is some fixed positive constant. Let us call such algorithm $\mathcal{A}_\varepsilon$. Consider the following algorithm $\mathcal{A}'$ that, for every input n-vertex split graph G runs as follows.

Step 1. Check if $n^\varepsilon < k$. If yes, then solve MinCDS on G by brute force. Otherwise, go to the next step;
Step 2. Run the reduction described previously on G to obtain G';
Step 3. Run $\mathcal{A}_\varepsilon$ on G' to obtain D';
Step 4. Compute D from D' (as explained in the reduction) and return D.

One may easily check that $\mathcal{A}'$ is a polynomial-time algorithm. We claim that $\mathcal{A}'$ always returns a $(1-\varepsilon^2)$-approximate CDS of G. If $\mathcal{A}'$ halts in step 1, then it returns an optimal CDS of G. Suppose now that $\mathcal{A}'$ halts after step 4. By hypothesis, we have that $|D'| \leq (1-\varepsilon)\ln|V(G')|\mathrm{OPT}_{\mathrm{MinCDS}}(G')$. But since $|D'| = |D|$, $|V(G')| \leq kn$, $n^\varepsilon \geq k$ and $\mathrm{OPT}_{\mathrm{MinCDS}}(G) = \mathrm{OPT}_{\mathrm{Min}k\text{-CDS}}(G')$, we conclude that

$$|D| \leq (1-\varepsilon)\ln n^{1+\varepsilon}\mathrm{OPT}_{\mathrm{MinCDS}}(G) = \left(1-\varepsilon^2\right)\ln n\mathrm{OPT}_{\mathrm{MinCDS}}(G).$$

Therefore, under the hypothesis that $\mathcal{NP} \not\subseteq \mathrm{DTIME}(n^{O(\log\log n)})$, the existence of $\mathcal{A}'$ contradicts Theorem 5, and so the result follows. $\square$

As we mentioned in the literature review subsection, Bonsma (2012) proved that MinCDS is $\mathcal{APX}$-complete on cubic graphs. The next theorem can be seen as a move in the direction of extending Bonsma's result. To make our presentation more self-contained, we define now the concept, which is used in the proof of Theorem 10, of an L-reduction.

Let P and Q be two optimization problems. An *L-reduction* from P to Q is a quadruple (f, g, α, β), where f and g are polynomial-time algorithms, and α and β are positive constants such that the following conditions hold:

(L1) if I is an instance of P, then $f(I)$ is an instance of Q;
(L2) $\mathrm{OPT}_Q(f(I)) \leq \alpha\,\mathrm{OPT}_P(I)$ for every instance I of P;
(L3) for every instance I of P and every feasible solution S to $f(I)$ with objective value $\mathrm{val}_Q(f(I), S)$, the algorithm g returns a solution $g(S)$ to I with objective value $\mathrm{val}_P(I, g(S))$ such that

$$|\mathrm{OPT}_P(I) - \mathrm{val}_P(I, g(S))| \leq \beta\,|\mathrm{OPT}_Q(f(I)) - \mathrm{val}_Q(f(I), S)|.$$

Theorem 10 *For every $k \in \mathbb{Z}_>$,* Min$k$-CDS *is $\mathcal{APX}$-complete on bipartite graphs of maximum degree* 4.

Proof We present an L-reduction from MinCVC to Min$k$-CDS. In fact, we show that the reduction described in the proof of Theorem 8 is an L-reduction. Then, we use the fact that MinCVC is $\mathcal{APX}$-hard on bipartite graphs of maximum degree 4 (Escoffier et al. 2010).

Let G be a bipartite graph with maximum degree 4, and let G' be the graph obtained from G using the reduction presented in the proof of Theorem 8. Observe that, for each CVC K of G, we can construct, in polynomial time, a k-CDS D of G' with $|D| \leq |K| + |E(G)|$ (see the proof of Theorem 8). Taking K as a minimum CVC of G, we can conclude that $\mathrm{OPT}_{\mathrm{Min}k\text{-CDS}}(G') \leq \mathrm{OPT}_{\mathrm{MinCVC}}(G) + |E(G)|$. (This inequality will be used in what follows.)

Since every vertex in G has degree at most 4, for every CVC K of G, it follows that $|E(G)| \leq 4|K|$. Therefore, for each CVC K of G, we can find, in polynomial time, a k-CDS D of G' such that $|D| \leq 5|K|$. Thus, $\mathrm{OPT}_{\mathrm{Min}k\text{-CDS}}(G') \leq 5\,\mathrm{OPT}_{\mathrm{MinCVC}}(G)$.

Conversely, given a k-CDS D of G', we know that, for each $e \in E(G)$, the set D contains w_e and at least one of the endpoints of e. Thus, if we take $K = D \cap V(G)$, we have that K is a CVC of G and $|K| \leq |D| - |E(G)|$. Therefore, $|K| - \mathrm{OPT}_{\mathrm{MinCVC}}(G) \leq |D| - |E(G)| - \mathrm{OPT}_{\mathrm{MinCVC}}(G) \leq |D| - \mathrm{OPT}_{\mathrm{Min}k\text{-CDS}}(G')$. This concludes the proof of the L-reduction.

Therefore, Min$k$-CDS is $\mathcal{APX}$-hard on bipartite graphs of maximum degree 4. By Corollary 2, this problem is in $\mathcal{APX}$, and therefore it is an $\mathcal{APX}$-complete problem. □

In the proof of Theorem 4, we showed that, for every graph G with poly-separators, there exists a $\sigma(G)$-approximation algorithm for MinWCDS, where $\sigma(G)$ is the cardinality of the largest minimal separator of G. We next prove that the approximation ratio of algorithm LPApproxMinWCDS, though simple as it is, is near-optimal. But before we do that, we need the following lemma, which can be considered a step towards a generalization of Theorem 2.

Lemma 2 *Let G be a graph and $\mathcal{S}_k(G)$ be the set of all minimal k-disruptive separators of G. A set $D \subseteq V(G)$ is a minimal k-*CDS *of G if and only if D is a minimal transversal of $\mathcal{S}_k(G)$.*

Proof Firstly, we prove that every k-CDS of G is a transversal of $\mathcal{S}_k(G)$. Let $D \subseteq V(G)$ be minimal a k-CDS of G and consider a k-disruptive separator Γ of G. We claim that $D \cap \Gamma \neq \emptyset$. Indeed, if $D \cap \Gamma = \emptyset$, then D is entirely contained in some component of $G - \Gamma$, a contradiction to the fact that Γ is a k-disruptive separator of G.

We next show that every minimal transversal of $\mathcal{S}_k(G)$ is a minimal k-CDS of G. Let $D \subseteq V(G)$ be a minimal transversal of $\mathcal{S}_k(G)$. Firstly, suppose to the contrary that $D[G]$ is not connected. Since D intersects every k-disruptive separator of G, there must be a component of $G[D]$, say C, such that $V(C)$ is a k-CDS of G; otherwise, $V(G)\backslash D$ would be a k-disruptive separator of G, contradicting the assumption on D. As we have shown before, $V(C)$ contains a transversal of $\mathcal{S}_k(G)$, but, since $V(C)$ is strictly contained in D, this contradicts the minimality of D. Thus, $D[G]$ is connected.

We now claim that D is a k-DS of G. Suppose to the contrary that there exists a vertex $v \in V(G)$ such that $v \notin N_k[D]$. Therefore, $N_k[v] \cap D = \emptyset$ and we conclude that v is not a k-universal vertex of G, that is, $\{v\}$ is not a k-CDS of G. Let us define $\Gamma = N_k(v)$. We claim that Γ is a k-disruptive separator of G. Clearly, $G - \Gamma$ is disconnected. Let C be a component of $G - \Gamma$. If $v \in V(C)$, then $V(C) = \{v\}$ and we already know that $\{v\}$ is not a k-CDS of G. If $v \notin V(C)$, then $v \notin N_k(V(C))$ and, once again, $V(C)$ is not a k-CDS of G. Hence, Γ is a k-disruptive separator of G, a contradiction to the fact that D is a transversal of $\mathcal{S}_k(G)$. This concludes the proof that D is a k-DS of G. Since $G[D]$ is also connected, D is a k-CDS of G. Finally, D is a minimal k-CDS of G because any k-CDS of G strictly contained in D would be a transversal of $\mathcal{S}_k(G)$, a contradiction the fact that D is a minimal transversal.

Let $D \subseteq V(G)$ be a minimal k-CDS of G. We have shown that D intersects every minimal k-disruptive separator of G. Now, we claim that D is a minimal transversal of $\mathcal{S}_k(G)$. Observe that any transversal of $\mathcal{S}_k(G)$ strictly contained in D would be a k-CDS of G, and this contradicts the fact that D is a minimal k-CDS of G. Therefore, D is a minimal transversal of $\mathcal{S}_k(G)$ and the result follows. □

So now let us prove, assuming $\mathcal{P} \neq \mathcal{NP}$, that the performance guarantee of algorithm LPApproxMinWCDS is close to the best we can hope for.

Theorem 11 *For $k \in \mathbb{Z}_>$ and a fixed integer $p \geq 2$, let $\mathcal{G}_p$ be the class of graphs G with $\sigma_k(G) = p$. For every $k \in \mathbb{Z}_>$ and every fixed $\varepsilon > 0$, if $\mathcal{P} \neq \mathcal{NP}$, then* Min$k$-CDS *cannot be approximated to within a factor of*

$$\max\left\{p - 1 - \varepsilon,\ 10\sqrt{5} - 21\right\},$$

on the class $\mathcal{G}_p$. Moreover, this claim holds even when we restrict it to $(1, 2)$-split graphs in $\mathcal{G}_p$.

Proof Let $\mathcal{H}$ be a hypergraph. We say that $\mathcal{H}$ is *simple* if none of its hyperedges is contained within another. Moreover, for every integer $p \geq 2$, we say that $\mathcal{H}$ is a

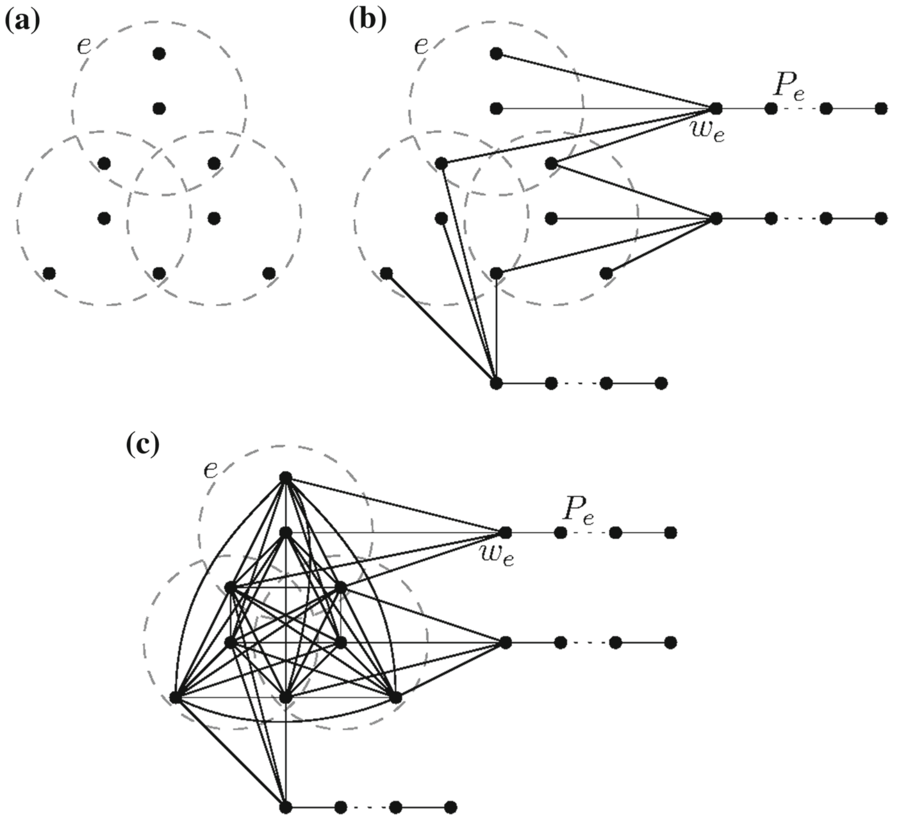

Fig. 6 Reduction from MINHVC to MINk-CDS: construction of G from the hypergraph $\mathcal{H}$. **a** A 4-uniform hypergraph $\mathcal{H}$ with 3 hyperedges (dashed circles). **b** Path P_e with k vertices and end vertex w_e. **c** Graph G. The vertices of the hypergraph $\mathcal{H}$ induce a clique in G

p-uniform hypergraph if all of its hyperedges have cardinality exactly p. A subset $K \subseteq V(\mathcal{H})$ is said to be a vertex cover of $\mathcal{H}$ if K intersects each hyperedge of $\mathcal{H}$. The *minimum hypergraph vertex cover problem* (MINHVC) consists in finding a vertex cover of $\mathcal{H}$ of minimum cardinality.

We show an L-reduction from MINHVC on simple p-uniform hypergraphs (fixed $p \geq 2$) to MINk-CDS on the class $\mathcal{G}_p$. We consider two cases.

Case 1: $p \geq 3$. Dinur et al. (2005) proved that, if $\mathcal{P} \neq \mathcal{NP}$, then for every $\varepsilon > 0$, MINHVC has no $(p-1-\varepsilon)$-approximation on p-uniform simple hypergraphs, for every $p \geq 3$. Let $\mathcal{H}$ be an n-vertex p-uniform simple hypergraph with $|E(\mathcal{H})| \geq 2$. From $\mathcal{H}$, we construct a graph G with $V(G) \supseteq V(\mathcal{H})$ as follows. For every $e \in E(\mathcal{H})$, we take a disjoint path P_e with k new vertices, and denote by w_e one of its end vertices. Then, we add an edge connecting w_e to every vertex in e. Finally, we make all vertices belonging to $V(\mathcal{H})$ pairwise adjacent in G, that is, $G[V(\mathcal{H})]$ is a clique. This construction is depicted in Fig. 6. Note that G is a $(1, 2)$-split graph that can be constructed in polynomial time in the size of $\mathcal{H}$.

Recall that $\mathcal{S}_k(G)$ denotes the set of all minimal k-disruptive separators of G. We claim that $\mathcal{S}_k(G) = E(\mathcal{H})$. It is easy to see that $E(\mathcal{H}) \subseteq \mathcal{S}_k(G)$. Now we prove that $\mathcal{S}_k(G) \subseteq E(\mathcal{H})$. For that, it suffices to show that, for every minimal k-disruptive separator Γ of G, there exists $e \in E(\mathcal{H})$ such that $\Gamma = N(w_e) \cap V(\mathcal{H})$. Consider $\Gamma \in \mathcal{S}_k(G)$. Note that, every minimal k-CDS of G is contained in $V(\mathcal{H})$. Therefore, due to the minimality of Γ, since, by definition, Γ intersects every minimal k-CDS of G, one may easily check that $\Gamma \subseteq V(\mathcal{H})$. Suppose, to the contrary, that $N(w_e) \cap V(\mathcal{H}) \neq \Gamma$ for all $e \in E(\mathcal{H})$. Since $N(w_e) \cap V(\mathcal{H}) \in \mathcal{S}_k(G)$ for every $e \in E(\mathcal{H})$, again, by the minimality of Γ, it follows that, for every $e \in E(\mathcal{H})$, $N(w_e) \cap V(\mathcal{H})$ is not strictly contained in Γ. Hence, $(N(w_e) \cap V(\mathcal{H}))\backslash\Gamma \neq \emptyset$ for every $e \in E(\mathcal{H})$. By the construction of G, we conclude that Γ is not a separator of G, a contradiction. Therefore, we have $\mathcal{S}_k(G) = E(\mathcal{H})$. Consequently, $\sigma_k(G) = p$, that is, $G \in \mathcal{G}_p$.

Now, we claim that G has a k-CDS of size q if and only if $\mathcal{H}$ has a vertex cover of size q. Consider a k-CDS D of G. We may assume that D is minimal and that $D \subseteq V(\mathcal{H})$. By Lemma 2, D is a transversal of $\mathcal{S}_k(G)$. Since $E(\mathcal{H}) = \mathcal{S}_k(G)$, we conclude that D is a vertex cover of $\mathcal{H}$. Conversely, let K be a vertex cover of $\mathcal{H}$. We may assume that K is minimal. Since $E(\mathcal{H}) = \mathcal{S}_k(G)$, it follows that K is a transversal of $\mathcal{S}_k(G)$. By Lemma 2, K is a k-CDS of G. Hence, we have $\mathrm{OPT}_{\mathrm{MIN}k\text{-CDS}}(G) = \mathrm{OPT}_{\mathrm{MINHVC}}(\mathcal{H})$, and this concludes the proof of the L-reduction.

Thus, every α-approximation for MINk-CDS on $(1, 2)$-split graphs in $\mathcal{G}_p$ yields an α-approximation for MINHVC on p-uniform simple hypergraphs. In view of the result shown by Dinur et al. (2005), we conclude that, if $\mathcal{P} \neq \mathcal{NP}$, for every $\varepsilon > 0$, there is no $(p - 1 - \varepsilon)$-approximation algorithm for MINk-CDS on $(1, 2)$-split graphs in the class $\mathcal{G}_p$.

Case 2: $p = 2$. In this case, we refer to the *minimum vertex cover problem* (MINVC), which is simply the restriction of MINHVC to graphs (which are 2-uniform hypergraphs); and for this problem, Dinur and Safra (2005) showed that MINVC has no $(10\sqrt{5} - 21)$-approximation if $\mathcal{P} \neq \mathcal{NP}$. In order to prove the result we claim, we use the same reduction discussed before, and from $\mathcal{H}$ (which is now a graph) we construct a $(1, 2)$-split graph G with $\sigma_k(G) = 2$. The result follows analogously, this time using the inapproximability result for MINVC.

The proof of the theorem is now complete, considering cases 1 and 2. □

7 Concluding remarks and future research

We studied the minimum weight k-hop connected dominating set problem, a generalization of the well-known minimum connected dominating set problem. We proved that, for every k, the decision version of MINk-CDS is $\mathcal{NP}$-complete on planar bipartite graphs of maximum degree 4 (showing that the hardness of MINCDS carries over to MINk-CDS on the same subclass of graphs).

We also proved a number of results on the (in)approximability of MINk-CDS. We showed that MINk-CDS is $\mathcal{APX}$-complete on the class of bipartite graphs with maximum degree 4 (we recall that it has been proven by Bonsma (2012) that MINCDS on cubic graphs is $\mathcal{APX}$-complete).

We showed that the inapproximability threshold $((1-\varepsilon)\ln n)$ of MINCDS that holds already for n-vertex split graphs (and also for bipartite graphs), proved by Chlebík and Chlebíková (2004), can be generalized to MINk-CDS on (1, 2)-split graphs (and also for bipartite graphs). We note here that, for $k \geq 2$, the last result does not hold for split graphs (on which the problem is trivial). We also showed an inapproximability threshold close to that of Chlebík and Chlebíková (2004) for MINCDS on the smaller class of n-vertex split graphs of diameter 2 (the smallest value of the diameter for which the problem is non-trivial).

On the positive side, we presented a type of meta-approximation theorem which says that, for every graph G, a $\beta(G)$-approximation for MINCDS on G can be turned into a $k\beta(G^k)$-approximation for MINk-CDS on G. As a consequence, we obtained an algorithm for MINk-CDS that, for every graph G and every fixed $0 < \epsilon \leq 1$, returns a $k(1+\epsilon)(\ln(\Delta(G^k)-1)+1)$-aproximate k-CDS of G. This result improves (asymptotically by a factor of 2) on the approximation originally proposed by Ren and Zhao (2011) for the minimum connected set cover problem (a generalization of MINk-CDS) which translates into a $2k(H(\Delta(G^k))+1)$-approximation for MINk-CDS. Furthermore, we showed two approximation algorithms for the weighted version of MINCDS restricted to graphs with polynomially many minimal separators, a class that includes, for instance, chordal graphs.

As future steps, we think it would be interesting to further investigate the graph parameter σ_k, possibly finding classes of graphs G for which this parameter is bounded, or graphs for which $\mathcal{S}_k(G)$ is polynomially bounded. Another line of research would be the design of better approximation algorithms for MINk-CDS, when $k \geq 2$, for special classes of graphs, such as the cubic graphs.

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